[Conference in the Memory of Prof. P.L. Hsu, July 2010]

Gradient estimates of Poisson equations on a Riemannian manifold and applications

Liming Wu

Academy of China and Université Blaise Pascal

1. Model and Questions Let

• $\mu = e^{-V(x)}dx/Z$ (Z is the normalization constant) with $V \in C^1$ on a complete connected Riemannian manifold M with convex or empty boundary ∂M

• the diffusion (X_t) generated by $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ (Δ, ∇ are respectively the Laplacian and the gradient on M) is μ-reversible and the corresponding Dirichlet form is given by

$$\mathcal{E}_\mu(g,g) = \int_M |\nabla g|^2 \, d\mu, \; g \in \mathbb{D}(\mathcal{E}_\mu) = H^1(\mathcal{X}, \mu)$$

If $\nu = f\mu$ with $0 < f \in C^1(M)$, then

$$I(\nu|\mu) = \int_M |\nabla \sqrt{f}|^2 \, d\mu = \frac{1}{4} \int_M \frac{|\nabla f|^2}{f} \, d\mu \tag{1}$$

is the Fisher-Donsker-Varadhan's information.

Question 1. Given g with $\mu(g) = 0$ (the heat source), the equilibrium heat distribution G satisfies the Poisson equation

$$-\mathcal{L}G = g.$$

How to estimate

$$\|G\|_{Lip} = \sup_{x \in M} |\nabla G|?$$

Question 2. To estimate the best constant $c_{P,1}$ in the L^1-Poincaré inequality

$$\int |f - \mu(f)| d\mu \le c_{P,1} \int |\nabla f| d\mu?$$

It is equivalent to (by Bobkov-Houdré (MAMS 97))

$$2\mu(A)[1 - \mu(A)] \le c_{P,1} \mu_\partial(\partial A), \ \forall A \subset M.$$

which is a variant of Cheeger's isoperimetric inequality.

Question 3. The Gaussian concentration constant c_G: the best constant such that for all g with $\|g\|_{Lip} \leq 1$,

$$\mathbb{P}_\beta\left(\frac{1}{t}\int_0^t g(X_s)\,ds \geq \mu(g) + r\right) \leq \left\|\frac{d\beta}{d\mu}\right\|_2 \exp\left(-t\frac{r^2}{2c_G}\right),\ \forall t, r > 0.$$

2. Gradient estimate : coupling method

Assume that the Ricci curvature is bounded from below by a constant $K \in \mathbb{R}$, i.e. $Ric_x(X,X) \geq K|X|^2$ for all $x \in M, X \in T_xM$ (the tangent space at x). Define for every real number $r \in [0, D]$,

$$a(r) = \inf\{\nabla_x \rho(x,y) \cdot \nabla V(x) + \nabla_y \rho(x,y) \cdot \nabla V(y);\ \rho(x,y) = r, y \notin cut(x \tag{2}$$

where $cut(x)$ is the cut-locus of x. Typically

$$a(r) \geq r \inf_{X \in TM, |X|=1} HessV(X,X), r > 0.$$

Let $b(r) : (-D/2, D/2) \to \mathbb{R}$ be an odd function such that for $r \in [0, D/2)$

$$b(r) = \begin{cases} -\sqrt{K(n-1)} \tan\left[r\sqrt{\frac{K}{n-1}}\right] - \frac{1}{2}a(2r), & \text{if } K \geq 0 \\ \sqrt{K^-(n-1)} \tanh\left[r\sqrt{\frac{K^-}{n-1}}\right] - \frac{1}{2}a(2r), & \text{if } K < 0. \end{cases} \tag{3}$$

Consider the one-dimensional generator on $(-D/2, D/2)$ with the Neumann boundary condition :

$$\mathcal{L}^{CW} = \frac{d^2}{dr^2} + b(r)\frac{d}{dr}.$$

Theorem 1 *(Chen-Wang 97) For the best Poincaré constant $c_P(\mu)$ in*

$$\int |f - \mu(f)|^2 d\mu \le c_P(\mu) \int |\nabla f|^2 d\mu$$

it holds that

$$c_P(\mu) \le c_P(\mathcal{L}^{CW}).$$

Method : coupling.

Assume now $Ric + HessV \ge \tilde{K}$. Consider

$$\tilde{b}(r) = \begin{cases} -\sqrt{\tilde{K}(n-1)}\tan\left[r\sqrt{\frac{\tilde{K}}{n-1}}\right], & \text{if } \tilde{K} \ge 0 \\ \sqrt{\tilde{K}^-(n-1)}\tanh\left[r\sqrt{\frac{\tilde{K}^-}{n-1}}\right], & \text{if } K < 0. \end{cases}$$

Theorem 2 *(Bakry-Qian 00) It holds that*

$$c_P(\mu) \le c_P(\mathcal{L}^{BQ})$$

where

$$\mathcal{L}^{BQ} = \frac{d^2}{dr^2} + \tilde{b}(r)\frac{d}{dr}.$$

Method: gradient estimate of the eigenfunction.

Theorem 3 (Wu 09 JFA) *Let $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ be defined on a connected complete Riemannian manifold M with empty or C^∞ smooth convex boundary ∂M, with the Neumann boundary condition, where $V \in C^2$ such that $\int_M \rho(x,o)^2 e^{-V(x)}dx < +\infty$. Assume that there is a sequence of convex relatively compact open domains M_n increasing to M such that ∂M_n is C^∞-smooth and convex, if M is non-compact.*

(a) Assume that

$$c_{Lip} := \int_0^{D/2} r \exp\left(\int_0^r b(s)ds\right) dr < +\infty. \tag{4}$$

Then the Poisson operator $(-\mathcal{L})^{-1}$ is bounded on $C_{Lip,0}(M)$ and its norm satisfies

$$\|(-\mathcal{L})^{-1}\|_{Lip} \leq c_{Lip} \leq \|(-\mathcal{L}^{CW})^{-1}\|_{Lip} \tag{5}$$

or equivalently if $-\mathcal{L}G = g$, then $\|G\|_{Lip} \leq c_{Lip}\|g\|_{Lip}$.

(b) For any bounded measurable function g with $\mu(g) = 0$, let G be a C^1 solution (in the distribution sense) of the Poisson equation $-\mathcal{L}G = g$. Then

$$\begin{aligned} &\sup_{x\in M} |\nabla G|(x) \le c_\delta \delta(g), \\ &c_\delta := \frac{1}{2}\int_0^{D/2} \exp\left(\int_0^r b(v)dv\right) dr = \frac{1}{4} m[-D/2, D/2] \end{aligned} \tag{6}$$

where $\delta(g) = \sup_{x\neq y} |g(x) - g(y)|$.

Remarks 1 There is a very rich theory about gradient estimates on Riemannian manifolds, often called Li-Yau's gradient estimates, see the book of Schoen-Yau for account of art and bibliographies.

3. Question 2.

Theorem 4 *(Wu JFA09) For the best constant $c_{P,1}$ in the L^1-Poincaré inequality,*

$$c_{P,1}(\mu) \leq 2c_\delta \leq c_{P,1}(\mathcal{L}^{CW}).$$

References : Buser, Yau, Ledoux etc.

4. Question 3.

Theorem 5 (Guillin-Léonard-Wu-Yao PTRF09) *Let $c_G > 0$ and let (X_t) be a μ-reversible and ergodic Markov process associated with $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$. The statements below are equivalent:*

(i) The following transportation-information $W_1 I(c)$ inequality holds true:

$$W_1^2(\nu, \mu) \le 2c_G I(\nu|\mu), \ \forall \nu; \qquad (W_1 I(c))$$

(ii) For all Lipschitz function g on M, $r > 0$ and $\beta \in M_1(M)$ such that $d\beta/d\mu \in L^2(\mu)$,

$$\mathbb{P}_\beta\left(\frac{1}{t}\int_0^t u(X_s)\,ds \ge \mu(u) + r\right) \le \left\|\frac{d\beta}{d\mu}\right\|_2 \exp\left(-\frac{tr^2}{2c_G\|g\|_{\mathrm{Lip}}^2}\right).$$

Theorem 6 $W_1 I(c_G)$ *holds with*

$$c_G \leq 2c_{Lip}^2.$$

Open Question. Whether $c_{LS}(\mu) \leq c_{LS}(\mathcal{L}^{CW})$?

Remarks 2

1. d_{ρ_a} is the metric associated with the carré-du-champ operator of the diffusion.

2. The quantity $C(\rho)$ in (**??**) is not innocent: Chen-Wang's variational formula for the spectral gap tells us that :

$$C_P(\mu) = \inf_{\rho} C(\rho, \rho).$$

Thanks!